"QUANTUM MECHANICS"

-Graduate Students-

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Lecturer Two

The Minimum Packet

The Minimum Uncertainty product is attained only, when two conditions are fulfilled :

(i) $\alpha\psi = \gamma\beta\psi$

(ii) $\int \psi^* (\alpha\beta + \beta\alpha)\psi \, dx = 0$

where $\alpha = x - \langle x \rangle$, $\beta = p - \langle p \rangle$.

The First Condition : (i)

$$(x - \langle x \rangle)\psi = \gamma (p - \langle p \rangle)\psi$$

$$= -i\hbar\gamma \frac{d\psi}{dx} - \gamma \langle p \rangle \psi$$

$$-i\hbar\gamma \frac{d\psi}{dx} = [x - \langle x \rangle + \gamma]\psi$$

$$\frac{d\psi}{\psi} = \left[\frac{i}{\gamma\hbar}(x - \langle x \rangle) + \frac{i}{\hbar}\langle p \rangle\right] dx$$

$$\psi(x) = N e^{\frac{i}{2\gamma\hbar}(x - \langle x \rangle)^2} e^{\frac{i}{\hbar}\langle p \rangle x} \qquad (16)$$

The second Condition (ii)

$$\int \psi^* (\alpha B) \psi \, dx + \int \psi^* (B\alpha) \psi \, dx = 0$$

$$\int (\alpha\psi)^* (B\psi) \, dx + \int (B\psi)^* (\alpha\psi) \, dx = 0$$

$$\gamma^* \int (B\psi)^* (B\psi) \, dx + \int (B\psi)^* (B\psi) \, dx = 0$$

$$(\gamma + \gamma^*) \underbrace{\int |B\psi|^2 \, dx}_{\neq 0} = 0$$

$$\Rightarrow \boxed{\gamma^* = -\gamma}$$, γ is a pure imagenery.

γ must be negative imagenery

$$\boxed{\gamma = -ic} \rightarrow \text{where } c \text{ is real +ve}$$

— (17)

Normalization:

$$\int_{-\infty}^{\infty} \psi^* \psi \, dx = 1$$

$$N^2 \int_{-\infty}^{\infty} e^{-\frac{1}{c\hbar}(x-\langle x\rangle)^2} \, dx = 1$$

$$\boxed{N = (\pi c \hbar)^{-1/4}} \quad \text{----(18)}$$

where $\int_{-\infty}^{\infty} u^{2n} \cdot e^{-\alpha u^2} \, du = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \alpha^{n+1/2}} \sqrt{\pi}$

$$= \sqrt{\frac{\pi}{\alpha}} \quad \text{for} \quad n = 0$$

where $n = 0, 1, 2, 3, \cdots$ ----- (19)

Constant C:

$$(\Delta x)^2 = \int \psi^* (x - \langle x\rangle)^2 \psi \, dx$$

$$= (\pi \hbar c)^{-1/2} \int (x - \langle x\rangle)^2 e^{-\frac{1}{\hbar c}(x-\langle x\rangle)^2} \, dx$$

H.W ⋮

$$N = \left(2\pi (\Delta x)^2\right)^{-1/4} = \frac{1}{\left(2\pi (\Delta x)^2\right)^{1/4}} \quad (11)$$

i The normalized minimum wave packet is:

$$\psi(x) = (2\pi(\Delta x)^2)^{-1/4} \; e^{\frac{-(x-\langle x\rangle)^2}{4(\Delta x)^2}} \; e^{i\frac{\langle p\rangle}{\hbar}x} \qquad \text{----(20)}$$

This Form salisfies the minimum uncertainty product $\Delta x \, \Delta p = \frac{1}{2}\hbar$

motion of free particle wave packet

Time evolution:

$$\psi(x,0) = (2\pi(\Delta x)^2)^{-1/4} e^{\frac{-x^2}{4(\Delta x)^2}} \quad \text{----- 21}$$

↳ Equation (20) with $\langle x\rangle = \langle p_x\rangle = 0$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,0)\, e^{-ikx}\, dx$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} (2\pi(\Delta x)^2)^{-1/4} \int e^{-\frac{x^2}{4(\Delta x)^2}} e^{-ikx}\, dx$$

$$= \text{-----} \int e^{-\left(\frac{x^2}{4(\Delta x)^2} + ikx\right)} dx$$

by completing the squeue:

$$-\left(\frac{x}{2\Delta x} + 2\Delta x\, \frac{ik}{2}\right)^2 - (\Delta x)^2 k^2$$

$$\therefore \phi(k) = \frac{[2\pi(\Delta x)^2]^{-1/4}}{\sqrt{2\pi}} e^{-(\Delta x)^2 k^2} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{2\Delta x} + ik\Delta x\right)^2} dx$$

Let $\frac{x}{2\Delta x} + ik\Delta x = u \Rightarrow du = \frac{1}{2\Delta x} dx$

$$\phi(k) = \cdots 2\Delta x \int_{-\infty}^{\infty} e^{-u^2} du = \cdots 2\Delta x\sqrt{\pi}$$

$$\vdots$$

$$\therefore \phi(k) = \left[\frac{2(\Delta x)^2}{\pi}\right]^{1/4} e^{-k^2(\Delta x)^2} \qquad \text{——(22)}$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\left[\frac{2(\Delta x)^2}{\pi}\right]^{1/4} \int_{-\infty}^{\infty} e^{-k^2(\Delta x)^2} e^{i(kx-\omega t)} dk$$

$$\omega = \frac{E}{\hbar} = \frac{p^2}{2m\hbar} = \frac{\hbar^2 k^2}{2m\hbar}$$

$$\therefore \psi(x,t) = \frac{1}{\sqrt{2\pi}}\left[\frac{2(\Delta x)^2}{\pi}\right]^{1/4} \cdot \int e^{-k^2(\Delta x)^2 + ikx - \frac{i\hbar}{2m}k^2 t} dk$$

$$= \cdots \int e^{-\left((\Delta x)^2 - \frac{i\hbar}{2m}t\right)k^2 + ikx} dk$$

Completing The squane :

$$-\left((\Delta x)^2 + \frac{i\hbar t}{2m}\right)\left[k^2 - \frac{ix}{(\Delta x)^2 + \frac{i\hbar t}{2m}}k\right]$$

$$= \left\{\left[k - \frac{ix}{2\left[(\Delta x)^2 + \frac{i\hbar t}{2m}\right]}\right]^2 + \frac{x^2}{4\left[(\Delta x)^2 + \frac{i\hbar t}{2m}\right]^2}\right\}$$

$$\vdots \quad \text{(H.W) Complete}$$

$$\psi(x,t) = (2\pi)^{-1/4}\left(\Delta x + \frac{i\hbar t}{2m\Delta x}\right)^{-1/2} e^{-\frac{x^2}{4(\Delta x)^2 + \frac{2i\hbar t}{m}}}$$

. Schiff eq.(12-20) --- (23)

The position probability density:

$$|\psi(x,t)|^2 = \left\{2\pi\left[(\Delta x)^2 + \frac{\hbar^2 t^2}{4m^2(\Delta x)^2}\right]\right\}^{-1/2} e^{-\frac{x^2}{2\left[(\Delta x)^2 + \frac{\hbar^2 t^2}{4m^2(\Delta x)^2}\right]}}$$

--- (24)

we have from eq.(21), at $t=0$

$$|\psi(x,0)|^2 = (2\pi(\Delta x)^2)^{1/2} e^{-\frac{x^2}{2(\Delta x)^2}} \quad \text{---(25)}$$

$(\Delta x)^2$ at $t=0$, is replaced by:

$$(\Delta x)^2 = (\Delta x)^2_0 + \frac{\hbar^2 t^2}{4m^2 (\Delta x)^2_0} \quad \text{—— (26)}$$

we have $\Delta x \Delta p = \frac{1}{2}\hbar$ (minimum Unc. product)

$$(\Delta x)^2_t = (\Delta x)^2_0 + \frac{(\Delta p)^2 t^2}{m^2} \quad \text{--- 27}$$

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